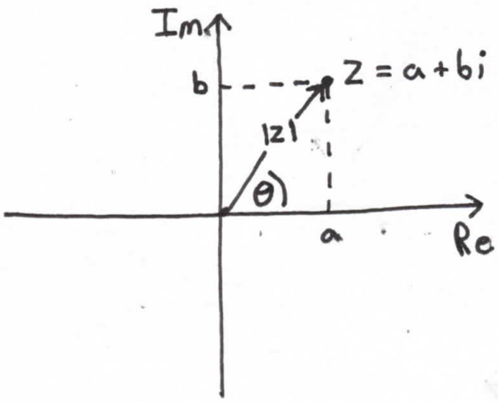
Ch 1 "Complex Numbers"

$Z = a + bi$, where z is a complex number, and a, b are Real numbers



$|z| = \sqrt{a^2+b^2}$

$\theta = \arctan(b/a)$

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Complex Conjugate: $\bar{z} = a - bi$

$\boxed{\text{Complex Exponentials}}$

$e^z = e^a \cdot e^{bi}$

$e^z = e^a[\cos(b) + i\sin(b)]$

Euler's Formula:

$e^{bi} = \cos(b) + i\sin(b)$

Euler's Identity:

$e^{\pi i} = -1$

$\boxed{\text{Complex Roots}}$ Abs val is unique. ~~A~~ argument is not

want to solve...

$z^n = w$, where w is a fixed complex number and we want the n^{th} root

$\downarrow$

$\rho^n e^{in\alpha} = re^{i\theta}$

$\boxed{\rho = \sqrt[n]{r}}$

$\therefore \boxed{z_k = \sqrt[n]{r} \cdot e^{i\frac{\theta + 2\pi k}{n}}}$, $0 < k < n-1$

$\boxed{\alpha = \frac{\theta + 2\pi k}{n}}$, where $0 < k < n-1$

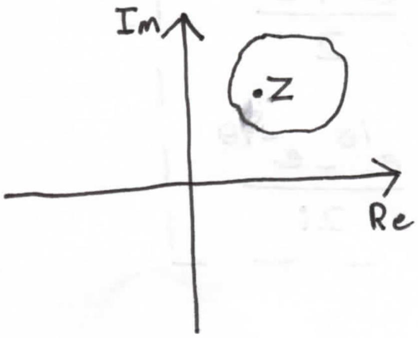
Ch 2 "Analytic Functions"

Complex Functions

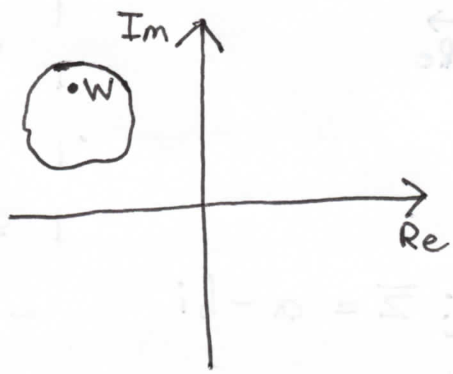
$W = f(z)$

Input, a complex number

Output, a complex number



Domain of f, set of all possible inputs



Range of f, set of all Possible outputs

$W = u(x,y) + iv(x,y)$

real functions

- x, y are real variables

Limits

$$\lim_{n\to\infty} z_n = z_0 \qquad \lim_{z\to z_0} f(z) = w_0$$

Analytic Function

- $f(z)$ is analytic on an open set G if it has a derivative that exist at every point of G

Cauchy-Riemann Equations

When $f(z)$ is differentiable at $z_0 = x_0 + iy_0$

$$f(z) = u(x,y) + iv(x,y)$$

$$f'(z_0) = \frac{du}{dx}(x_0,y_0) + \frac{dv}{dx}(x_0,y_0)\,i$$

$$(1)\ \frac{du}{\partial x} = \frac{dv}{dy} \qquad (2)\ \frac{du}{dy} = -\frac{dv}{dx}$$

$$= \frac{du}{dx}(x_0,y_0) - \frac{du}{dy}(x_0,y_0)\,i$$

- For $f(z)$ to de differentiable at $z_0 = x_0 + iy_0$, The Cauchy-Riemman Equn's must hold

- For $f(z)$ to be analytic. These equn's must hold at every point on the open-set G

EXERCISES

Section 2.1

Q1 (a) $f(z) = 3z^2 + 5z + i + 1$

$= 3(x+yi)^2 + 5(x+yi) + i + 1$

$= 3(x^2 + 2xyi - y^2) + 5(x+yi) + i + 1$

$= 3x^2 + 6xyi - 3y^2 + 5x + 5yi + i + 1$

$= \boxed{3x^2 + 5x - 3y^2 + 1 + i(6xy + 5y + 1)}$

Harmonic Functions:

- $\dfrac{d^2 u}{dx^2} + \dfrac{d^2 u}{dy^2} = 0$
- And Both these Derivatives (above) are Continuous

~~Stat~~

Statements:

- If $f = u + iv$ is analytic, u, v are harmonic
- If u is harmonic, there is a v such that $f = u + iv$ is analytic

Analytic Continuation: (If function is Analytic)

$$f(z) = u(x,y) + i v(x,y)$$

- To get in terms of z set $y = 0$, change $x \to z$

2.5

$\boxed{Q1}$ (a) $f(z) = z^2 + 2z + 1$

$z = x + iy$

Laplace eqn: $\dfrac{du^2}{dx^2} + \dfrac{d^2 u}{dy^2} = 0$

$$f(z) = (x+iy)^2 + 2(x+iy) + 1$$
$$= \cancel{x^2} + 2xyi - \cancel{y^2} + \cancel{2x} + 2yi + \cancel{1}$$
$$= x^2 - y^2 + 2x + 1 + i(2xy + 2y)$$

$$u(x,y) = x^2 - y^2 + 2x + 1$$

$\dfrac{du}{dx} = 2x + 2$ ¦ $\dfrac{du}{dy} = -2y$

$\dfrac{du^3}{dx^2} = 2$ ¦ $\dfrac{du^3}{dy^2} = -2$

$$v(x,y) = 2xy + 2y$$

$\therefore$ Real Satisfies

Ch 3 "Elementary Functions"

3.1

Complex Polynomial: $P_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots a_n z^n$

- Rational Functions are just ratios of polynomials

ie: $R_{m,n}(z) = \dfrac{a_0 + a_1 z + a_2 z^2 + \cdots a_m z^m}{b_0 + b_1 z + b_2 z^2 + \cdots b_n z^n}$

(Gauss)

Thereom 1:

- Every <u>Non-Constant</u> polynomial with complex coefficients has at least one zero in $\mathbb{C} \rightarrow$ Complex Plane

- Therefore, a polynomial of degree "n" has n zeros

3.2

<u>The Complex Exponential:</u> $e^z = e^x(\cos y + i \sin y)$

$|e^z| = e^x$, $\frac{d}{dz} e^z = e^z$, This function is "entire" (except ∞)

$\arg(e^z) = y + 2\pi n$

- $e^{z_1} = e^{z_2}$ <u>Iff</u>: $z_1 = z_2 + 2\pi n$

- Trig Identities that hold from normal calculus page 113

Page 113

Trigometric Identities:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\frac{d}{dz}\sin(z) = \cos(z) \qquad \frac{d}{dz}\cos(z) = -\sin(z)$$

Hyperbolic functions:

$$\sinh(z) = \frac{e^{z} - e^{-z}}{2} \qquad \cosh(z) = \frac{e^{z} + e^{-z}}{2}$$

$$\frac{d}{dz}\sinh(z) = \cosh(z) \qquad \frac{d}{dz}\cosh(z) = \sinh(z)$$

$\boxed{3.3}$ $W = \log(z)$ if $z = e^{W}$

- the log function is the inverse of the exponential function

$$\log(z) = \ln|z| + i\,\underbrace{\arg(z)}_{\text{multi-valued}}$$

Principal Branch:

$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z)$, - Remember $\mathrm{Arg}(z)$ is $(-\pi, \pi]$

$\frac{d}{dz}\mathrm{Log}(z) = \frac{1}{z}$, Not defined on *or at $z=0$ the non-positive real-axis (ie: no imaginary part)

- log Identities $\boxed{\text{Pg } 119}$

f(z) is a branch of log(z) iff $\boxed{e^{f(z)} = z}$

$\boxed{3.5}$ Complex Power Function:

$$f(z) = z^{\alpha} = e^{\alpha \log(z)}$$

Principal value:

$$z^{\alpha} = e^{\alpha \operatorname{Log}(z)}$$

Ex: $\alpha = \frac{1}{n} \rightarrow f(z) = z^{\frac{1}{n}} = e^{\frac{1}{n}\log(z)} = e^{\frac{1}{n}\cdot[\ln|z| + i\arg(z)]}$

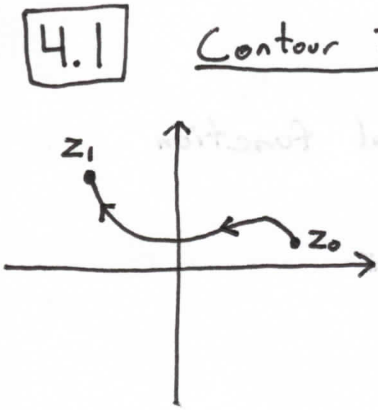
$$= e^{\frac{1}{n}\cdot \ln|z|} \cdot e^{\frac{1}{n}\cdot \arg(z)\cdot i}$$

$$= \sqrt[n]{|z|} \cdot e^{\frac{\arg(z)}{n}\cdot i} \longrightarrow$$

*Notice this is the complex roots formula. As expected for $z^{1/n}$

Ch 4 "Complex Integration"

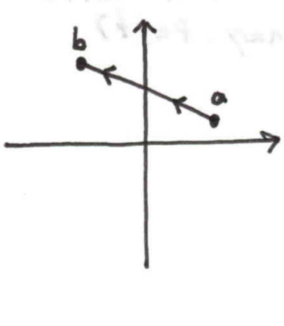
$\boxed{4.1}$ Contour Integrals:



$$\int_{\gamma} f(z)\,dz = \int_a^b f(\gamma(t))\cdot \gamma'(t)\,dt$$

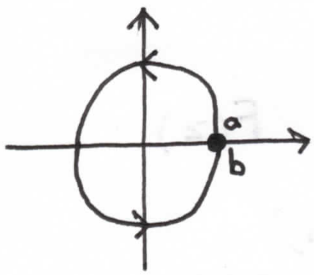
- where $\gamma: [a,b] \rightarrow \mathbb{C}$ "Contour in the complex plane"

Parameterization: Ex: "Straight line"



$$\gamma(t) = a + t(b-a)\ ;\ 0 \leq t \leq 1$$

Parameterization: Ex: "Unit Circle"



$$\gamma(t) = e^{it}; \quad t \in [0, 2\pi]$$

Types of Curves:

Smooth arc

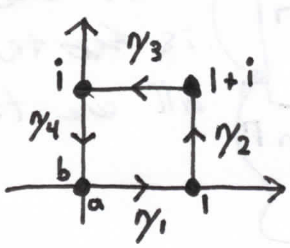
Smooth closed curve

Not smooth

Not Smooth

Arc Length $\boxed{\text{pg. 159}}$

Parameterization: Ex: "Positive unit square"



$$\gamma_1(t) = t \quad [0,1]$$
$$\gamma_2(t) = 1 + ti \quad [0,1]$$
$$\gamma_3(t) = (1-t) + i \quad [0,1]$$
$$\gamma_4(t) = (1-t)i \quad [0,1]$$

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz$$

$\boxed{4.2}$ Independence of Path:

- If γ is a contour from $z_0 \to z_1$ and $F(z)$ is anlaytic on γ with $F'(z) = f(z)$. Then,

$$\int_\gamma f(z)\,dz = F(z_1) - F(z_0)$$

- Where $F(z)$ is the anti-derivative of $f(z)$

$\boxed{4.3}$

Thereom:

- Let $f(z)$ be an analytic function in D

① $f(z)$ has an anti-derivative $F(z)$ in D

② $\boxed{\int_\Gamma f(z)\,dz = 0 \text{ for any closed contour } \Gamma}$

*Must be analytic on Γ

③ $\int_\Gamma f(z)\,dz = F(z_1) - F(z_0)$

*If one is ~~for~~ true all are true

$$\int_\Gamma z^n = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}$$

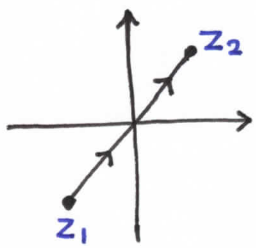
*where Γ is the positively oriented unit-circle

$\boxed{4.4}$ Cauchy's Integral Formula (General):

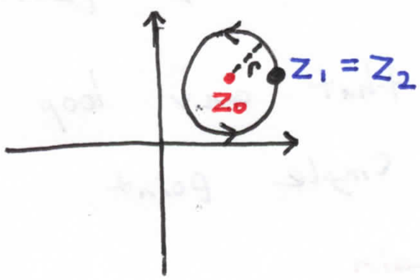
- If γ is a simple, ~~cotas~~ closed contour (*positively oriented) and if $f(z)$ is analytic on γ and inside γ. Then,

$$\oint_{\gamma} \frac{f(z)}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} \cdot f^{(n)}(z_0)$$

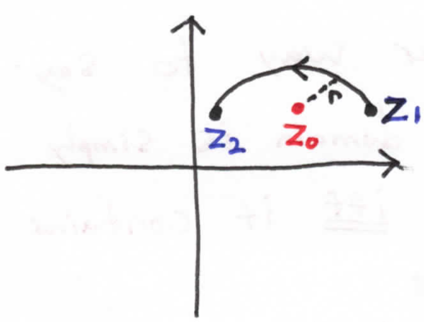
Parameterizations:



$$z(t) = z_1 + t(z_2 - z_1);\ t \in [0,1]$$

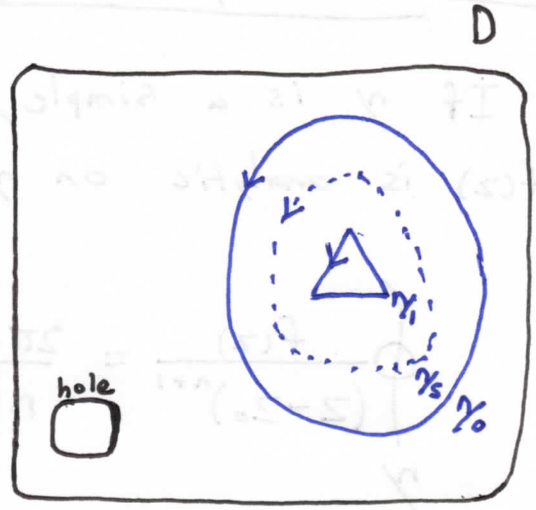


$$z(t) = z_0 + re^{it};\ t \in [0, 2\pi]$$



$$z(t) = z_0 + re^{it};\ t \in [0, \pi]$$

Deformation of Contours:



– a loop γ_0 is continuously deformable to the loop γ_1 in the domain D if there exists a function $z(s,t)$ continuous for $s \in [0,1]$, $t \in [0,1]$ that satisfies:

① For each intermediary contour, s in $[0,1]$, $z(s,t)$ parameterizes a loop γ_s in D

② $z(0,t)$ parameterizes γ_0

③ $z(1,t)$ parameterizes γ_1

Simply connected Domain:

– ~~Any domain D ... that can be deformed to a single point~~

– Any domain D, that has the property that any loop in this domain can be deformed to a single point is known as a Simply Connected Domain

$\gamma_0 = z_0(t),\ t \in [0,1]$

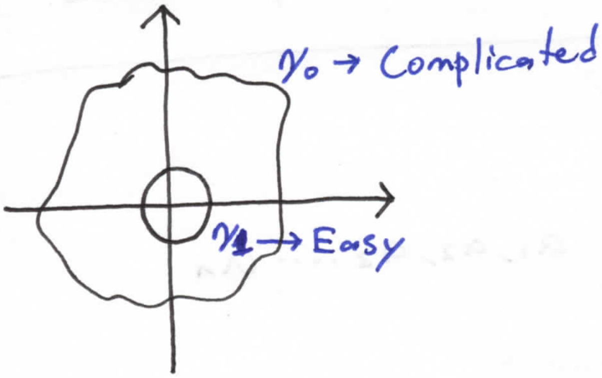
* Another way to say this. A domain is simply Connected <u>iff</u> it contains no holes

$$z(s,t) = (1-s)\cdot z_0(t),\ t \in [0,1]$$

$$z(0,t) = z_0(t)$$

$$z(1,t) = 0$$

Deformation Invariance Thereom:



- If $f(z)$ is an analytic function in D containing loops γ_0, γ_1. ~~Then~~ ~~Then~~ And they can be continuously deformed. Then,

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$$

$\boxed{4.5}$ Louisville's Thereom:

- If $f(z)$ is an entire function. ie: $f(z)$ is analytic is all of $\mathbb{C}$ and $f(z)$ is bounded, ie: There exists a real number M such that $|f(z)| \leq M$ for all z. Then,

• $f(z)$ must be a constant

Maximum Modulus Principle (MMP):

Cauchy Estimate:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w-z)^{n+1}}\,dw$$

Ch 5 "Series Representation for Analytic Functions"

5.1 Sequence:

- A sequence of complex numbers $a_1, a_2, a_3 \dots a_n$

$\lim_{n\to\infty} a_n$ → Converges to a
→ Diverges

Geometric Sequence:

$a_n = x_n + y_n i$

$a_n = c^n$

$$\lim_{n\to\infty} a_n = \begin{cases} 0, & \text{if } |c|<1 \\ 1, & \text{if } c = 1 \\ \text{DNE} & \text{in all other cases} \end{cases}$$

Infinite Series (General):

$$\sum_{j=0}^{\infty} c_j = c_0 + c_1 + c_2 + c_3 \dots\dots + c_n$$

Geometric Series:

$\sum_{j=0}^{\infty} z^j$
- Converges if $|z|<1$
- Diverges if $|z|\geq 1$

- If $|z|<1$, $\lim_{n\to\infty} S_n = \frac{1}{1-z}$

$\therefore \sum_{j=0}^{\infty} z^j = \frac{1}{1-z}$; if $|z|<1$

The n^{th} Term Test:

If $\sum_{j=0}^{\infty} c_j$ Converges. Then,

- $\lim_{n\to\infty} c_n = 0$

- This is a $\boxed{\text{test for divergence}}$. ie: if $\lim_{n\to\infty} c_n \neq 0$ then the series diverges.

- However, if $\lim_{n\to\infty} c_n = 0$, the series may converge or diverge

Comparison Test:

- We have $\sum_{j=0}^{n} c_j$ as an infinite series.

- If we can find M_j, where $|c_j| \leq M_j$ for every $j \geq N$. $(M_N, M_{N+1}, M_{N+2} \ldots\ldots)$ AND $\sum_{j=N}^{\infty} M_j$ Converges. Then,

- $\sum_{j=0}^{\infty} c_j$ Converges as well

Ratio Test:

- If $c_j \neq 0$ for all j AND $\lim\limits_{n \to \infty} \frac{|c_{n+1}|}{|c_n|} = L$. Then,

- $\sum\limits_{j=0}^{\infty} c_j$ $\boxed{\text{Converges}}$ if $L < 1$

- $\sum\limits_{j=0}^{\infty} c_j$ $\boxed{\text{Diverges}}$ if $L > 1$

*Note, if $\lim\limits_{n \to \infty} \frac{|c_{n+1}|}{|c_n|} = 1$, or DNE: Then,

- The Ratio Test is Inconclusive

Uniform Convergence:

Taylor Series:

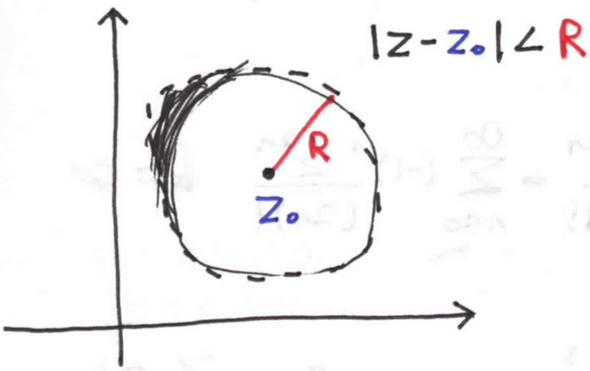
$$\boxed{a_n = \frac{f^{(n)}(z_0)}{n!}}$$

- If $f(z)$ is analytic around z_0. Then,

$$\bullet \sum_{n=0}^{\infty} \frac{f^{n}(z_0)}{n!} \cdot (z-z_0)^n = f(z_0) + \frac{f'(z_0)}{1!} \cdot (z-z_0) + \frac{f''(z_0)}{2!} \cdot (z-z_0)^2 \ldots\ldots$$

This is called the Taylor Series for $f(z)$ around z_0

Thereom: Taylor Series:



- If $f(z)$ is analytic on $|z-z_0|<R$,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{n}(z_0)}{n!} \cdot (z-z_0)^n \text{; for all}$$

z "inside" this Circle with radius R

- The radius of Convergence R, is the distance from z_0 to the <u>Closest</u> Singularity

* Special Case → If $z_0 = 0$. It is also known as the Mauclarin Series

Power Series:

(Geometric Series)

Radius of Convergence:

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 \dots + z^n = \sum_{n=0}^{\infty} z^n \qquad R = 1$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad R = \infty$$

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \frac{z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \qquad R = \infty$$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \frac{z^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \qquad R = \infty$$

$$\operatorname{Log}(z) = \frac{(z-1)}{1} - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} + \dots \frac{(z-1)^n}{n}$$

$z_0 = 1$
Converges in the disk $|z-1| < 1$

Determining Taylor Series Expansion of a function:

- Rarely use the actual definition. ie: Find $f'(z_0), f''(z_0) \dots$
- Instead, manipulate a known power series (above) into looking like the function given, and use that expansion
- Find the Disk of convergence by looking from z_0 to the <u>closest</u> singularity

Cauchy Multiplication:

Ex: "Find the first four terms of the Taylor Series Expansion of the function $f(z) = \frac{e^z}{1-z}$ at $z=0$

- Instead of finding $f(0), f'(0), f''(0), f'''(0)$ and using the formula for Taylor Series. We can use Cauchy Multiplication.

$$f(z) = g(z)h(z) \quad \text{where,} \quad g(z) = e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

$$\sum_{k=0}^{\infty} a_k z^k \cdot \sum_{k=0}^{\infty} b_k z^k$$

$$[a_0 + a_1 z + a_2 z^2 \dots] \cdot [b_0 + b_1 z + b_2 z^2 \dots] \quad h(z) = \frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots$$

$$\boxed{= a_0 b_0 + (a_1 b_0 + a_0 b_1) z + (a_2 b_0 + a_1 b_1 + a_0 b_2) z^2 + (a_3 b_0 + a_2 b_1 + a_1 b_2 + a_0 b_3) z^3 \dots}$$

$$f(z) = g(z)h(z) = \left[1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots\right] \cdot \left[1 + z + z^2 + z^3 \dots\right]$$

$$= 1 + (1+1)z + (1+1+\tfrac{1}{2})z^2 + (1+1+\tfrac{1}{2}+\tfrac{1}{6})z^3 + \dots$$

$$= \boxed{1 + 2z + \frac{5}{2} z^2 + \frac{8}{3} z^3 + \text{"Higher order Terms"}}$$

- e^z is analytic in all $\mathbb{C}$, therefore $R = \infty$

- $\frac{1}{1-z}$ is analytic in $|z| < 1$, therefore $R = 1$

- We go for the smaller of the two, Therefore $f(z)$ converges in the disk $|z| < 1$

Laurent Series:

- If $f(z)$ is analytic on $R_1 < |z-z_0| < R_2$. Then,

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n \cdot (z-z_0)^n}_{\text{analytic part}} + \underbrace{\sum_{n=1}^{\infty} a_{-n} \cdot (z-z_0)^{-n}}_{\text{principal part}}$$; for all z on $R_1 < |z-z_0| < R_2$

analytic part: All positive powers of $(z-z_0)$

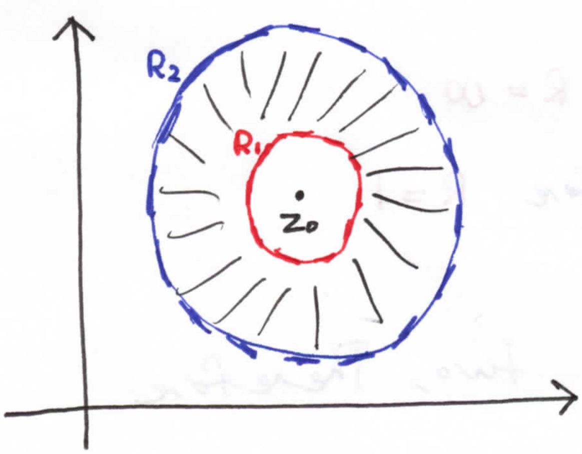
principal part: All ~~no~~ negative powers of $(z-z_0)$

Determining Coefficients (a_n):

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s-z_0)^{n+1}}\, ds$$; $n = 0, \pm 1, \pm 2, \pm 3 \ldots$

C is a closed curve inside the annulus surrounding z_0

Annulus (Donut):



Ch 6 The Residue Thereom

5.6

Definition (Zeros):

- z_0 is a zero of order n if,
  - $f(z_0) = f'(z_0) = f''(z_0), \ldots\ldots = f^{n-1}(z_0) = 0$ but $f^{n}(z_0) \neq 0$
- Essentially, a Zero z_0 is a value of z so that $f(z_0) = 0$

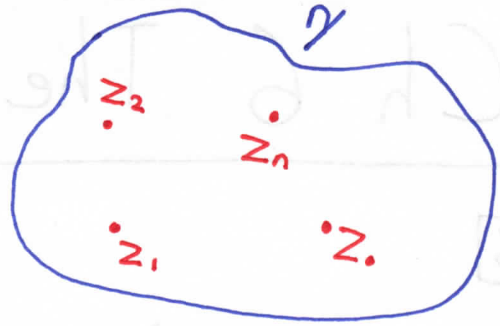
Definition (Poles):

- z_0 is a pole of order n if,
  - $\lim\limits_{z \to z_0} (z-z_0)^n \cdot f(z) \neq 0 \longleftrightarrow f(z) = \dfrac{g(z)}{(z-z_0)^n}$, $g(z_0) \neq 0$
- Essentially, a pole is a value of z so that $f(z_0) = \dfrac{g(z_0)}{0}$, so that $f(z) \to \infty$ as $z \to z_0$

Key Notes:

- if z_0 is a zero of order n for $f(z)$, then z_0 is a pole of order n for $1/f(z)$
- If $f(z)$ and $g(z)$ have Zeros / Poles of order m and n respectively at $z = z_0$, then,
  - $h(z) = f(z)g(z)$ has a Zero / Pole of order $(m+n)$ at $z = z_0$

Cauchy's Residue Thereom



Isolated Singularity (3 Cases):

- z_0 is an isolated singularity of $f(z)$ if $f(z)$ is analytic inside γ given by $0 < |z - z_0| < R$

Laurent Series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

• **Removable Singularity:**

- There are no negative powers of ~~the~~ $(z - z_0)^{-n}$

Ex: $f(z) = \dfrac{\sin(z)}{z}$

- Removable Singularity at $z = 0$

• **Pole:** (of order m. Simple if $m = 1$)

- If m is the largest positive integer such that $a_{-m} \neq 0$.

Ex: $f(z) = \dfrac{1}{(z - 3i)^7}$

- Pole of order 7 at $z = 3i$

• **Essential Singularity:**

- If there are an infinite number of negative powers of $(z - z_0)^n$

Ex: $f(z) = e^{1/z}$

- Essential Singularity at $z = 0$

Residue Thereom:

- If γ is a simple closed positively oriented contour and $f(z)$ is analytic on and inside γ except at some isolated singularities $z_0, z_1, z_2 \ldots\ldots z_n$. Then,

- $$\int_{\gamma} f(z)\,dz = 2\pi i \sum_{j=1}^{n} \text{Res}(z_j)$$

  Residue of $f(z)$ at z_j

Residue:

- A residue of $f(z)$ at an isolated singularity z_0 is the coefficient a_{-1} in the Laurent series around z_0

$$f(z) = \ldots\ldots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)^1} + a_0 + a_1(z-z_0)^1 + a_2(z-z_0)^2 \ldots\ldots$$

How to determine Residue at z_0:

① Write out the Laurent Series around z_0. Grab a_{-1} term

② (If z_0 is a Simple Pole)

- $$\text{Res}(z_0) = \lim_{z \to z_0} (z-z_0) \cdot f(z)$$

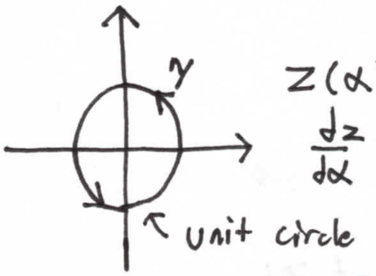
③ (If z_0 is a Pole of order m)

- $$\text{Res}(z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m \cdot f(z)\right]$$

Evaluating Real Integrals

*If you get a complex answer, you've made a mistake

Trigonometric (Proper Integral $[0, 2\pi]$):



$$z(\alpha) = e^{i\alpha};\ [0, 2\pi]$$

$$\frac{dz}{d\alpha} = ie^{i\alpha}$$

$$dz = ie^{i\alpha}\, d\alpha$$

$$\therefore d\alpha = \frac{dz}{ie^{i\alpha}} = \boxed{\frac{dz}{iz}}$$

Note:

- $\cos(\alpha) = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha}) = \frac{1}{2}(z + 1/z)$
- $\sin(\alpha) = \frac{1}{2i}(e^{i\alpha} - e^{-i\alpha}) = \frac{1}{2i}(z - 1/z)$

Improper Integral $[-\infty, \infty]$: